On the First Cohomology Group of a Minimal Set (Symposium on Dynamical Systems)

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Notations and Definitions

Let (Y,ρ_t) be a flow on a compact metric space Y .

(i) The flow (Y,ρ_t) is said to be a minimal flow on Y , if every orbit is dense in Y .

(ii) A subset Σ of Y is said to be a local section if it satisfies: (a) $h : \overline{\Sigma}\times(-\mu,\mu) \to \{\rho_t(y) \mid y\in\overline{\Sigma},\ -\mu < t < \mu\}$ defined by $h(y,t) = \rho_t(y)$ for some $\mu > 0$ (such is called a collar-size for Σ), and (b) $\{\rho_t(y) \mid y\in\Sigma,\ t\in J\}$ is open for any open J R . Moreover if Σ is compact, then we call it a global section.

(iii) $\bar{H}^*(Y)$ denotes the Alexander cohomology of Y with the real coefficients. For a presheaf Γ of modules on Y , $\check{H}^*(Y;\Gamma)$ denotes the Čech cohomology with the coefficient Γ .

1. Preliminaries

At the meeting last year, I have reported the following results. (For the precise proof, see [1].)

PROPOSITION 1. For a minimal flow (M,ξ_t) and a local section Σ , we can construct a minimal flow $(\tilde{M},\zeta_t)$ with the following properties: (a) $\tilde{M}$ is a compact metric space, (b) there is a continuous map $p : \tilde{M} \to M$ such that $p\circ\zeta_t = \xi_t\circ p$, (c) $\overline{p^{-1}(\Sigma)}$ is a global section of $(\tilde{M},\zeta_t)$, and (d) $\overline{p^{-1}(\Sigma)}$ is totally disconnected; i.e., $\dim(\overline{p^{-1}(\Sigma)}) = 0$.

Let (M,ξ_t) be a minimal flow and Σ be a local section with a collar-size μ. And let $(\tilde{M},\zeta_t)$ be a minmal flow which is constructed in the previous proposition. Define X to be $X = M - \{\xi_t(X) \mid x \in \Sigma, -\mu < t < 0\}$, and $\tilde{X}$ to be $\tilde{X} = \{\zeta_t(\tilde{x}) \mid \tilde{x} \in p^{-1}(\Sigma), -\mu < t < 0\}$. Let Γ_j $(j = 1, 2, 3)$ be presheaves defined by $\Gamma_1(U) = \bar{H}^0(U)$, $\Gamma_2(U) = \bar{H}^0(p^{-1}(U))$ and $\Gamma_3(U) = \mathrm{Coker}(P^*)$ where p^* is the homomorphism $\bar{H}^0(U) \to \bar{H}^0(p^{-1}(U))$ induced by $p : p^{-1}(U) \to U$. Then we have

PROPOSITION 2. There is an exact sequence

$$\check{H}^0(x;\Gamma_2) \to \check{H}^0(X;\Gamma_3) \to \bar{H}^1(X) \to 0 .$$

2. Results

Using the exact sequence in Proposition 2, we can give a method for calculating the first cohomology of a 3-dimensional minimal set.

In what follows, (M,ξ_t) will be a minimal flow on a 3-dimensional comapact manifold which is generated by a C^1-vector field.

Notations

(a) For a real valued function F defined on a subset D of M, $\hat{F}$ denotes a map $\hat{F} : D \to M$ defined by $\hat{F}(x) = \xi_{F(x)}(x)$.

(b) Let Σ be a local section, then we use the following notations.

$T_\Sigma : M \to R$ defined by $T_\Sigma(x) = \inf\{t > 0 \mid \xi_t(x) \in \overline{\Sigma}\}$,

$B^1_\Sigma \subset \partial\Sigma : B^1_\Sigma = \{x \in \partial\Sigma \mid \hat{T}_\Sigma(x) \in \partial\Sigma\}$,

$B^j_\Sigma \subset \partial\Sigma : B^j_\Sigma = \{x \in \partial\Sigma \mid \hat{T}_\Sigma(x) \in B^{j-1}_\Sigma\}$ $(j = 2, 3, \ldots)$

$A^j_\Sigma \subset \Sigma : A^j_\Sigma = \{x \in \Sigma \mid \hat{T}_\Sigma(x) \in B^j_\Sigma\}$ $(j = 1, 2, 3, \ldots)$

$C_\Sigma \subset \Sigma : C_\Sigma = \{x \in \Sigma \mid \hat{T}_\Sigma(x) \in \partial\Sigma\}$.

Let Σ be a local section of (M, ξ_t) which is homeomorphic to a 2-disk. Here we make an assumption.

Assumption I. $A_\Sigma^j = \phi$ for $j \geqq 2$, and A_Σ^1 is a finite set.

Let $A_\Sigma^1 = \{a_1, a_2, \ldots, a_N\}$ consist of N-points. We denote by $C_1, C_2, \ldots, C_{2N}$ the components of $C_\Sigma \setminus A_\Sigma^1$. (It is easy to see that if A_Σ^1 consists of N-points, then $C_\Sigma \setminus A_\Sigma^1$ has 2N connected components.) Then, for each point a_k of A_Σ^1, we can take a neighborhood $S_k \subset \Sigma$ of a_k with the properties: (a) there are continuous functions $\sigma_{k,j}$ (j = 1, 2, 3) such that $\hat{\sigma}_{k,j}(S_k) \subset \Sigma'$ (j = 1, 2), $\hat{\sigma}_{k,3}(S_k) \subset \Sigma$, and $\hat{\sigma}_{k,j}(a_k) = \hat{T}_\Sigma^j(a_k)$ (j = 1, 2, 3), where Σ' is a local section which includes the closure of Σ. We make another assumption on Σ.

Assumption II. $S_k \cap (C_\Sigma \setminus A_\Sigma^1)$ has exactly three components $\gamma_{k,j}$ (j = 1, 2, 3) such that $\hat{\sigma}_{k,2}(\gamma_{k,1}) \subset \Sigma$, $\hat{\sigma}_{k,2}(\gamma_{k,2}) \cap \overline{\Sigma} = \phi$, and $\hat{\sigma}_{k,2}(\gamma_{k,3}) \subset \partial\Sigma$.

REMARK. We can show that there is a local section which satisfies the Assumptions I and II.

Fixing a numbering of the components of A_Σ^1 and $C_\Sigma \setminus A_\Sigma^1$, for each k ($1 \leqq k \leqq N$), we define integers $k(j)$ (j = 1, 2, 3, 4 and $1 \leqq k(j) \leqq 2N$) so that $C_{k(j)} \cap \gamma_{k,j} \neq \phi$ (j = 1, 2, 3) and $\hat{T}_\Sigma(a_k) \in \overline{C}_{k(j)}$. And a $2N \times 2N$ matrix $\Lambda_\Sigma = [\lambda_1, \lambda_2, \ldots, \lambda_{2N}]$ (λ_j is a 2N-vector) is defined by

$$\begin{aligned}(u_1, u_2, \ldots, u_{2N})\lambda_{2k-1} &= u_{k(1)} - u_{k(2)} \\ (u_1, u_2, \ldots, u_{2N})\lambda_{2k} &= u_{k(2)} - u_{k(3)} + u_{k(4)}\end{aligned} \qquad (k = 1, \ldots, 2N).$$

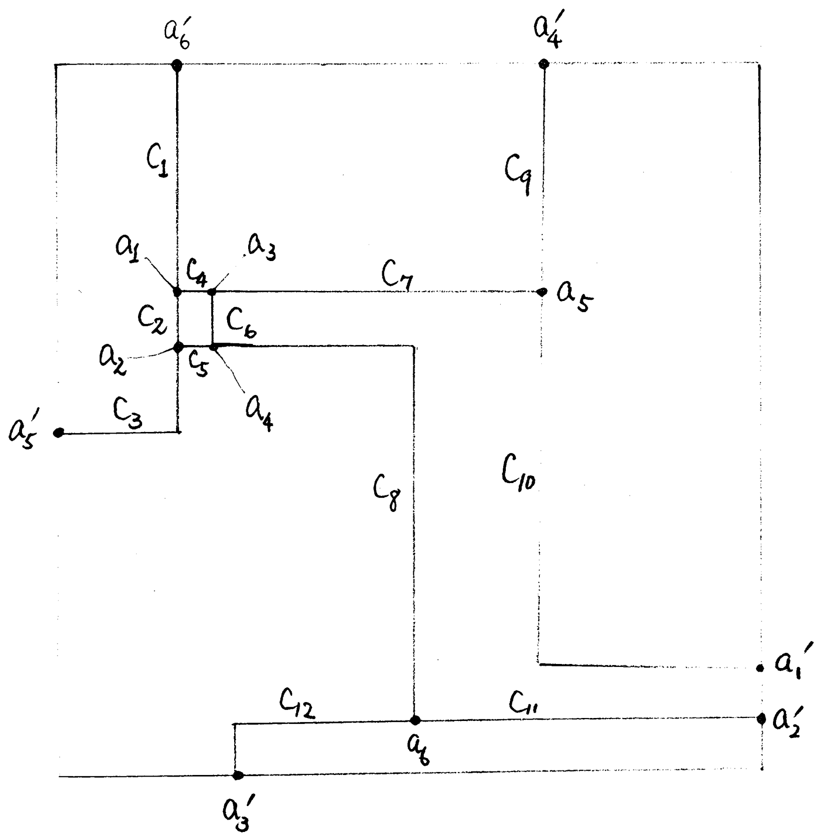
Our result is the following

<u>THEOREM</u>. $\bar{H}^1(M)$ is isomorphic to the solution space of the equation $u\Lambda_\Sigma = 0$.

For the proof, see [1].

3. An Example

We consider a flow on the 3-torus $T^3 = R^3/Z^3$ which is generated by a vector field $-(\partial/\partial x) - \sqrt{2}(\partial/\partial y) - \sqrt{3}(\partial/\partial z)$. As a local section we take $\Sigma = \{(x,y,z) \mid x = 0 , 0 < y, z < 1/2\}$. Then C_Σ appears as the following figure.



$a'_k = \hat{T}_\Sigma(a_k)$

Numbering the components of A_Σ^1 and $C_\Sigma \setminus A_\Sigma^1$ as in the figure, we have

$$
\begin{array}{llll}
c_{1(1)} = c_1 , & c_{1(2)} = c_2 , & c_{1(3)} = c_4 , & c_{1(4)} = c_{10} , \\
c_{2(1)} = c_3 , & c_{2(2)} = c_2 , & c_{2(3)} = c_5 , & c_{2(4)} = c_{11} , \\
c_{3(1)} = c_7 , & c_{3(2)} = c_4 , & c_{3(3)} = c_6 , & c_{3(4)} = c_{12} , \\
c_{4(1)} = c_8 , & c_{4(2)} = c_5 , & c_{4(3)} = c_6 , & c_{4(4)} = c_9 , \\
c_{5(1)} = c_9 , & c_{5(2)} = c_{10} , & c_{5(3)} = c_7 , & c_{5(4)} = c_3 , \\
c_{6(1)} = c_{12} , & c_{6(2)} = c_{11} , & c_{6(3)} = c_8 , & c_{6(4)} = c_1 .
\end{array}
$$

Hence the equation $u\Lambda_\Sigma = 0$ becomes as follows:

$$
\begin{array}{ll}
u_1 - u_2 = 0 , & u_2 - u_4 + u_{10} = 0 , \\
u_3 - u_2 = 0 , & u_2 - u_5 + u_{11} = 0 , \\
u_7 - u_4 = 0 , & u_4 - u_6 + u_{12} = 0 , \\
u_8 - u_5 = 0 , & u_5 - u_6 + u_9 = 0 , \\
u_9 - u_{10} = 0 , & u_{10} - u_7 + u_3 = 0 , \\
u_{12} - u_{11} = 0 , & u_{11} - u_8 + u_1 = 0 .
\end{array}
$$

One can easily see that this equation has three independent solutions. Therefore, using the Theorem, we get $\bar{H}^1(T^3) \simeq R^3$.

REFERENCE

[1] Ishii, I., On the first cohomology group of a minimal set, to appear in Tokyo Journal of Mathmatics Vol.1 No.1.